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Asymptotically linear elliptic systems in $\mathbb{R}^N$

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ABSTRACT

In this paper, we show that semilinear elliptic systems of the form

$$\begin{cases} -(\Delta u - u) - \mu(\Delta v - v) = g(v), \\ -(\Delta v - v) - \lambda(\Delta u - u) = f(u), \end{cases} \quad x \in \mathbb{R}^N$$

possess at least one nontrivial solution pair $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, where λ and μ are positive numbers, $f(t)$ and $g(t)$ are continuous functions on $\mathbb{R}$ and asymptotically linear as $t \to +\infty$.

1. Introduction

In this paper, we are concerned with the following nonlinear elliptic systems

$$\begin{cases} -(\Delta u - u) - \mu(\Delta v - v) = g(v), \\ -(\Delta v - v) - \lambda(\Delta u - u) = f(u), \end{cases} \quad x \in \mathbb{R}^N, \tag{1.1}$$

where λ and μ are nonnegative numbers, $N \geqslant 3$, $f(t)$ and $g(t)$ are continuous functions on $\mathbb{R}$ and asymptotically linear as $t \to \infty$. We are interested in the existence of nontrivial weak solution pairs $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Definition 1.1. $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ is said to be a weak solution pair of problem (1.1) if (u, v) satisfies

$$\int_{\mathbb{R}^N} (\nabla u \cdot \nabla \psi + u\psi + \mu \nabla v \cdot \nabla \psi + \mu v \psi + \nabla v \cdot \nabla \varphi + v\varphi + \lambda \nabla u \cdot \nabla \varphi + \lambda u \varphi)\, dx$$
$$- \int_{\mathbb{R}^N} f(u)\varphi\, dx - \int_{\mathbb{R}^N} g(v)\psi\, dx = 0$$

for all $(\varphi, \psi) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

If both λ and μ are equal to zero, much attention has been paid to the existence and multiplicity of nontrivial solution pairs of problem (1.1) in bounded domain, see, for example, [1,3,7,9,19] and the references therein. But for system like (1.1) in $\mathbb{R}^N$ with $\lambda = \mu = 0$, there are fewer results. De Figueiredo and J. Yang [8] show the existence of a radial solution pair under the assumption that $f(x, t)$ and $g(x, t)$ are radially symmetric with respect to x. Later, [18] extended the results of [8]. Another approach to the existence of nontrivial solution pairs of system of the type

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$$-\Delta u = \frac{\partial H(x,u,v)}{\partial v}, \qquad -\Delta v = \frac{\partial H(x,u,v)}{\partial u}$$

with $u, v \in H^1(\mathbb{R}^N)$, where $H(x,u,v) = -q(x)uv + \bar{H}(x,u,v)$ with $q(x) \to +\infty$ as $|x| \to \infty$ and $\bar{H}$ is superlinear or sublinear as $(u^2+v^2)^{\frac{1}{2}}$ appeared in [6]. In the works [6,8,18], a basic assumption is required, that is, f and g satisfy Ambrosetti–Rabinowitz condition. So if $f(x,t)$ and $g(x,t)$ are asymptotically linear as $t \to \infty$, then the Ambrosetti–Rabinowitz condition is not satisfied, and it is hard to show that Palais–Smale sequences of the energy functional are uniformly bounded. One then turns to Cerami condition. In [12], G. Li and J. Yang had considered the following asymptotically linear elliptic systems

$$\begin{cases} -\Delta u + u = g(x,v), \\ -\Delta v + v = f(x,u), \end{cases} \quad x \in \mathbb{R}^N.$$

They obtained a positive solution by using a Linking theorem, a Pohozaev type identity and concentration-compactness principle of P.L. Lions [11] under the Cerami condition.

Let $E = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. For problem (1.1), the associated functional is

$$\begin{aligned} I(z) = I(u,v) = \int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla v + uv + \frac{\lambda}{2}|\nabla u|^2 + \frac{\mu}{2}|\nabla v|^2 + \frac{\lambda}{2}|u|^2 + \frac{\mu}{2}|v|^2 \right) dx \\ - \int_{\mathbb{R}^N} F(u)\,dx - \int_{\mathbb{R}^N} G(v)\,dx. \end{aligned} \tag{1.2}$$

The quadratic part

$$Q(z) = Q(u,v) = \int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla v + uv + \frac{\lambda}{2}|\nabla u|^2 + \frac{\mu}{2}|\nabla v|^2 + \frac{\lambda}{2}|u|^2 + \frac{\mu}{2}|v|^2 \right) dx$$

is positively definite or indefinite in E depending on the range of $\lambda\mu$. Actually, $Q(z)$ is indefinite in E if $0 \leqslant \lambda\mu < 1$ and the functional $I(z)$ possesses geometry of Linking type, if $\lambda\mu > 1$, $Q(z)$ is positively definite and $I(z)$ possesses Mountain Pass geometry. In [14], C. Peng and J. Yang had considered problem (1.1) in bounded domain with $0 < \lambda\mu < 1$, by using Linking theorem under the Cerami condition, relative Morse index and Liouville theorem, they got a nontrivial solution pair with superlinear f and g. They also considered problem (1.1) with $0 < \lambda\mu < 1$ in bounded domain [15] or in $\mathbb{R}^N$ [16] when f and g are asymptotically linear.

In the present paper, we will discuss problem (1.1) in E under the assumptions that $f(t)$ and $g(t)$ are asymptotically linear as $t \to \infty$ and λ, μ satisfy $\lambda\mu > 1$, which allows us to define an equivalent norm on E. In fact, let E be equipped with the norm

$$\|z\|_E = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2 + |\nabla v|^2 + |v|^2\right) dx \right)^{\frac{1}{2}},$$

where $z = (u,v)$. Since $\lambda\mu > 1$, then there exists a real number $l > 0$ such that $\lambda > l > \frac{1}{\mu}$ and we have

$$\begin{aligned} &\max\left\{ \frac{1+\lambda}{2}, \frac{1+\mu}{2} \right\} \left(|\nabla u|^2 + |u|^2 + |\nabla v|^2 + |v|^2\right) \\ &\quad \geqslant \nabla u \cdot \nabla v + uv + \frac{\lambda}{2}\left(|\nabla u|^2 + u^2\right) + \frac{\mu}{2}\left(|\nabla v|^2 + v^2\right) \\ &\quad \geqslant \min\left\{ \frac{\lambda - l}{2}, \frac{\mu}{2} - \frac{1}{2l} \right\} \left(|\nabla u|^2 + |u|^2 + |\nabla v|^2 + |v|^2\right). \end{aligned} \tag{1.3}$$

Then we may introduce a new inner product on E by the formula

$$\langle (u,v), (\varphi,\psi) \rangle = \int_{\mathbb{R}^N} (\lambda \nabla u \cdot \nabla \varphi + \lambda u\varphi + \nabla u \cdot \nabla \psi + u\psi + \mu \nabla v \cdot \nabla \psi + \mu v\psi + \nabla v \cdot \nabla \varphi + v\varphi)\,dx,$$

and the corresponding norm is

$$\|z\| = \left(\langle z,z \rangle\right)^{\frac{1}{2}} = \left\{ \int_{\mathbb{R}^N} \left[\lambda\left(|\nabla u|^2 + u^2\right) + 2(\nabla u \cdot \nabla v + uv) + \mu\left(|\nabla v|^2 + v^2\right)\right] dx \right\}^{\frac{1}{2}}$$

for any $z = (u,v) \in E$. The norms $\|\cdot\|$ and $\|\cdot\|_E$ are then equivalent if $\lambda\mu > 1$ by (1.3).

We assume that f and g satisfy

(H_1) $f, g \in C(\mathbb{R}, \mathbb{R})$.
(H_2) $\lim_{|t|\to 0}(f(t)/t) = \lim_{|t|\to 0}(g(t)/t) = 0$.
(H_3) $\lim_{|t|\to\infty}(f(t)/t) = l > 0$, $\lim_{|t|\to\infty}(g(t)/t) = m > 0$.
(H_4) $f(t)/t$ and $g(t)/t$ are non-decreasing in $|t|$.
(H_5) $\frac{1}{2}tf(t) - F(t) > 0$ and $\frac{1}{2}tg(t) - G(t) > 0$ for any $t \neq 0$, where $F(t) = \int_0^t f(s)\,ds$, $G(t) = \int_0^t g(s)\,ds$.

Our main results are as follows.

Theorem 1.1. *Suppose* (H_1)–(H_4) *hold. If* $\lambda\mu > 1$, $l > \lambda + 1$ *and* $m > \mu + 1$, *then the problem* (1.1) *possesses at least one nontrivial solution pair* $z = (u, v) \in E$.

Theorem 1.2. *Suppose* (H_1)–(H_5) *hold. If* $\lambda\mu > 1$, $l > \lambda + 1$ *and* $m > \mu + 1$, *then the problem* (1.1) *possesses a nontrivial least energy solution pair* $z = (u, v) \in E$.

Our conditions on $f(t)$ and $g(t)$ for the existence of a nontrivial solution pair for problem (1.1) are in some sense standard. Many standard results for single equation with asymptotically linear term $f(x,t)$

$$-\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N$$

have been obtained under similar assumptions on $f(x,t)$. See [5,10,13] and the references therein.

In order to prove Theorems 1.1 and 1.2, we use the frame work in [13]. We will prove Theorem 1.1 by finding critical points of the corresponding functional (1.2), which are weak solution pairs of (1.1). We first prove that $I(z)$ possesses Mountain Pass type geometry and use Mountain Pass Theorem under the Cerami compactness condition [4] to get a (C_c)-sequence $\{z_n\} \subset E$ of $I(z)$. Note that the usual Mountain Pass Theorem with (*PS*) compactness condition is not good enough to deal with asymptotically linear problem. The main difficulty consists in that one could not prove that any $(PS)_c$-sequence is bounded in E without Ambrosetti–Rabinowitz condition in general. Next, we will prove that $(C)_c$-sequence $\{z_n\} \subset E$ for $I(z)$ is bounded. During this process, the concentration-compactness principle of P.L. Lions [11] is involved. We then show that $\{z_n\}$ has a subsequence which converges weakly to a nontrivial critical point of $I(z)$ by the concentration-compactness principle. Hence Theorem 1.1 will be proved. Using the concentration-compactness principle again, we finally show that

$$I^\infty = \inf\bigl\{I(z){:}\ I'(z) = 0,\ z = (u, v) \in E \setminus \{0\}\bigr\}$$

is assumed by some $z_0 = (u_0, v_0)$ with $u_0 \not\equiv 0$, $v_0 \not\equiv 0$. Hence Theorem 1.2 will be proved.

In Section 2, we give some preliminary results. The main results (Theorems 1.1 and 1.2) will be proved in Section 3.

2. Preliminary results

Suppose in this section that λ, μ satisfy $\lambda\mu > 1$. From assumptions (H_1)–(H_3), we see that there is a $2 < p < 2N/(N-2)$ and that for any $\epsilon > 0$, there is a $c_\epsilon > 0$ such that for all $t \in \mathbb{R}$,

$$\bigl|f(t)\bigr|, \bigl|g(t)\bigr| \leqslant \epsilon|t| + c_\epsilon|t|^{p-1}, \qquad \bigl|F(t)\bigr|, \bigl|G(t)\bigr| \leqslant \epsilon|t|^2 + c_\epsilon|t|^p. \tag{2.1}$$

So the corresponding energy functional I defined in (1.2) is well defined on E and of class $C^1(E, \mathbb{R})$. Moreover, the Fréchet derivative I' satisfies

$$\begin{aligned}\bigl\langle I'(u,v), (\varphi, \psi)\bigr\rangle = {} & \int\limits_{\mathbb{R}^N} (\nabla u \cdot \nabla\psi + u\psi + \mu\nabla v \cdot \nabla\psi + \mu v\psi + \nabla v \cdot \nabla\varphi + v\varphi + \lambda\nabla u \cdot \nabla\varphi + \lambda u\varphi)\,dx \\ & - \int\limits_{\mathbb{R}^N} f(u)\varphi\,dx - \int\limits_{\mathbb{R}^N} g(v)\psi\,dx\end{aligned} \tag{2.2}$$

for all $(\varphi, \psi) \in E$.

A sequence $\{z_n\} \subset E$ is called a Cerami sequence of a C^1 functional I on E at level c ($(C)_c$-sequence for short), if $I(z_n) \to c$ and $(1 + \|z_n\|)I'(z_n) \to 0$ in E^* as $n \to \infty$, see [2]. To get a $(C)_c$-sequence, we use the following Mountain Pass Theorem in [4].

Proposition 2.1. *Let E be a real Banach space with its dual space E^* and suppose that $I \in C^1(E, \mathbb{R})$ satisfies the condition*

$$\max\bigl\{I(0), I(z_1)\bigr\} \leqslant \alpha < \beta \leqslant \inf_{\|z\|=\rho} I(z)$$

for some $\alpha < \beta$, $\rho > 0$ and $z_1 \in E$ with $\|z_1\| > \beta$. Let $c \geqslant \beta$ be characterized by

$$c = \inf_{\gamma\in\Gamma}\max_{0\leqslant\tau\leqslant 1} I\bigl(\gamma(\tau)\bigr),$$

where $\Gamma = \{\gamma \in C([0,1], E)\colon \gamma(0) = 0,\ \gamma(1) = z_1\}$ *is the set of continuous paths joining* 0 *and* z_1. *Then, there exists a sequence* $\{z_n\} \subset E$ *such that*

$$I(z_n) \to c \geqslant \beta \quad \text{and} \quad \big(1 + \|z_n\|\big)\big\|I'(z_n)\big\|_{E^*} \to 0.$$

Lemma 2.1. *For the functional* I *defined by* (1.2), *if the conditions* (H_1) *and* (H_4) *hold, and for any* $\{z_n\} \subset E$ *with*

$$\big\langle I'(z_n), z_n\big\rangle \to 0 \quad \text{as } n \to \infty,$$

then there is a subsequence, still denoted by $\{z_n\}$, *such that*

$$I(tz_n) \leqslant \frac{1+t^2}{2n} + I(z_n) \quad \text{for all } t \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

In particular, if $\langle I'(z_n), z_n\rangle = 0$ *for all* $n \geqslant 1$, *then for any* $t > 0$,

$$I(tz_n) \leqslant I(z_n).$$

Proof. This lemma is essentially due to [13] for the single equation, for the reader's convenience, we give its proof here. By the assumption, we may assume that there exists a subsequence, still denoted by $\{z_n\}$, such that for all $n \geqslant 1$, $-\frac{1}{n} < \langle I'(z_n), z_n\rangle < \frac{1}{n}$. Denoting $z_n = (u_n, v_n)$, then we have

$$-\frac{1}{n} + \int_{\mathbb{R}^N} \big[f(u_n)u_n + g(v_n)v_n\big]\,dx < \|z_n\|^2 < \int_{\mathbb{R}^N} \big[f(u_n)u_n + g(v_n)v_n\big]\,dx + \frac{1}{n}. \tag{2.3}$$

So by (2.3), for any $t > 0$,

$$\begin{aligned} I(tz_n) &= \frac{t^2}{2}\|z_n\|^2 - \int_{\mathbb{R}^N} \big[F(tu_n) + G(tv_n)\big]\,dx \\ &\leqslant \frac{t^2}{2n} + \int_{\mathbb{R}^N} \left[\frac{1}{2} f(u_n)u_n t^2 - F(tu_n)\right] dx + \int_{\mathbb{R}^N} \left[\frac{1}{2} g(v_n)v_n t^2 - G(tv_n)\right] dx. \end{aligned}$$

Set $h_1(t) = \frac{1}{2}t^2 sf(s) - F(ts)$, $h_2(t) = \frac{1}{2}t^2 sg(s) - G(ts)$, by conditions (H_1) and (H_4), it is easy to see that

$$h_1'(t) = f(s)ts - f(ts)s = \begin{cases} \geqslant 0 & \text{if } t \leqslant 1; \\ \leqslant 0 & \text{if } t \geqslant 1, \end{cases}$$

$$h_2'(t) = g(s)ts - g(ts)s = \begin{cases} \geqslant 0 & \text{if } t \leqslant 1; \\ \leqslant 0 & \text{if } t \geqslant 1, \end{cases}$$

which mean that $h_1(t) \leqslant h_1(1)$ and $h_2(t) \leqslant h_2(1)$ for all $t > 0$. So

$$I(tz_n) \leqslant \frac{t^2}{2n} + \int_{\mathbb{R}^N} \left[\frac{1}{2} f(u_n)u_n - F(u_n)\right] dx + \int_{\mathbb{R}^N} \left[\frac{1}{2} g(v_n)v_n - G(v_n)\right] dx.$$

On the other hand, by (2.3) again, we have

$$\begin{aligned} I(z_n) &= \frac{1}{2}\|z_n\|^2 - \int_{\mathbb{R}^N} \big[F(u_n) + G(v_n)\big]\,dx \\ &\geqslant \frac{-1}{2n} + \int_{\mathbb{R}^N} \left[\frac{1}{2} f(u_n)u_n - F(u_n)\right] dx + \int_{\mathbb{R}^N} \left[\frac{1}{2} g(v_n)v_n - G(v_n)\right] dx, \end{aligned}$$

that is

$$\int_{\mathbb{R}^N} \left[\frac{1}{2} f(u_n)u_n - F(u_n)\right] dx + \int_{\mathbb{R}^N} \left[\frac{1}{2} g(v_n)v_n - G(v_n)\right] dx \leqslant \frac{1}{2n} + I(z_n).$$

Therefore,

$$I(tz_n) \leqslant \frac{1+t^2}{2n} + I(z_n).$$

Similarly, if $\langle I'(z_n), z_n\rangle = 0$ for all $n \geqslant 1$, then for any $t > 0$, $I(tz_n) \leqslant I(z_n)$. $\square$

Lemma 2.2. *Let* (H_1)–(H_3) *hold, then we have*:

(a) *There exist* $\rho, \beta > 0$ *such that* $I(z) \geqslant \beta$ *for all* $z \in E$ *with* $\|z\| = \rho$;
(b) *There exist* $z_1 \in E$ *with* $\|z_1\| > \beta$ *such that* $I(z_1) < 0$ *if* $m > 1 + \mu$, $l > 1 + \lambda$.

Proof. (a) It follows from (2.1) and the Sobolev embedding theorem that, for any $\epsilon > 0$, there is a $c_\epsilon > 0$ such that

$$\int_{\mathbb{R}^N} F(u)\,dx + \int_{\mathbb{R}^N} G(v)\,dx \leqslant c\epsilon\|z\|^2 + c_\epsilon\|z\|^p$$

for all $z = (u, v) \in E$. This and (1.2) imply (a).

(b) Following [13] or [17], we denote

$$\big(d(N)\big)^2 = \int_{\mathbb{R}^N} e^{-2|x|^2}dx, \qquad D(N) = 4\big(d(N)\big)^{-2}\int_{\mathbb{R}^N} |x|^2 e^{-2|x|^2}\,dx.$$

For $\alpha > 0$, let $z_\alpha = (w_\alpha, w_\alpha)$, where $w_\alpha(x) = (d(N))^{-1}\alpha^{\frac{N}{4}}e^{-\alpha|x|^2}$ for $x \in \mathbb{R}^N$. Then $w_\alpha(x) \in H^1(\mathbb{R}^N)$ and

$$|w_\alpha|_2 = 1, \qquad \big|\nabla w_\alpha(x)\big|_2^2 = 4\alpha^2\int_{\mathbb{R}^N} |x|^2 w_\alpha(x)^2\,dx = \alpha D(N).$$

Since

$$\|z_\alpha\|^2 = \langle z_\alpha, z_\alpha\rangle = (\lambda + \mu + 2)\int_{\mathbb{R}^N} \big(|\nabla w_\alpha|^2 + w_\alpha^2\big)\,dx,$$

and $m > 1 + \mu$, $l > 1 + \lambda$, so if we choose $\alpha \in (0, \frac{l+m-(\lambda+\mu+2)}{D(N)(\lambda+\mu+2)})$, then

$$|\nabla w_\alpha|_2^2 < \frac{l + m - (\lambda + \mu + 2)}{(\lambda + \mu + 2)}|w_\alpha|_2^2. \tag{2.4}$$

So by (2.4) and Fatou's lemma, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{I(tz_\alpha)}{t^2} &= \frac{1}{2}\|z_\alpha\|^2 - \lim_{t\to\infty}\bigg[\int_{\mathbb{R}^N}\frac{F(tw_\alpha)}{t^2}\,dx + \int_{\mathbb{R}^N}\frac{G(tw_\alpha)}{t^2}\,dx\bigg]\\ &\leqslant \frac{1}{2}\|z_\alpha\|^2 - \frac{1}{2}(l + m)|w_\alpha|_2^2\\ &= \frac{1}{2}(\lambda + \mu + 2)\int_{\mathbb{R}^N} |\nabla w_\alpha|^2\,dx - \frac{1}{2}\big[l + m - (\lambda + \mu + 2)\big]\int_{\mathbb{R}^N} w_\alpha^2\,dx < 0,\end{aligned}$$

which implies that $I(tz_\alpha) < 0$ with $\|tz_\alpha\| > \beta$ when t is large enough. □

To verify that the $(C)_c$-sequence of I is bounded in E, we use the following proposition, which was proved in [15].

Proposition 2.2. *Suppose that* l, m *are two positive constants and* $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ *is a weak solution pair of the linear system*

$$\begin{cases} -(\Delta u - u) - \mu(\Delta v - v) = mv, \\ -(\Delta v - v) - \lambda(\Delta u - u) = lu, \end{cases} \quad x \in \mathbb{R}^N. \tag{2.5}$$

Then $u = v = 0$.

Next we deal with the boundedness of $(C)_c$-sequences.

Lemma 2.3. *Suppose* (H_1)–(H_4) *hold. If* $\{z_n\}$ *is a* $(C)_c$*-sequence* $(c > 0)$ *of* I, *then* $\{z_n\}$ *is bounded in* E.

Proof. Following [13], we argue indirectly. Assume that $\|z_n\| \to \infty$ as $n \to \infty$. Let $t_n = \frac{2\sqrt{c}}{\|z_n\|}$,

$$w_n = t_n z_n = (t_n u_n, t_n v_n) \overset{\Delta}{=} \big(w_n^1, w_n^2\big), \qquad \rho_n(x) = \big|w_n(x)\big|^2 = t_n^2\big(u_n^2 + v_n^2\big).$$

By Lions' concentration-compactness principle [11], for the concentration functions of $\rho_n(x)$, either vanishing happens:

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int\limits_{y+B_R}\rho_n(x)\,dx=0$$

for any $0<R<+\infty$, or nonvanishing happens: there exist $\eta>0$, $R<+\infty$ and $\{y_n\}\subset\mathbb{R}^N$ such that

$$\liminf_{n\to\infty}\int\limits_{y_n+B_R}\rho_n(x)\,dx\geqslant\eta>0.$$

We shall show both cases would lead to a contradiction, hence, $\{z_n\}$ is bounded.

If vanishing occurs, by the vanishing lemma [11], we would have $w_n^1\to 0$, $w_n^2\to 0$ in $L^q(\mathbb{R}^N)$ for $2<q<2N/(N-2)$ as $n\to\infty$. Then by (2.1), we see that

$$\int\limits_{\mathbb{R}^N}F\big(w_n^1\big)\,dx\to 0\quad\text{and}\quad\int\limits_{\mathbb{R}^N}G\big(w_n^2\big)\,dx\to 0.$$

So

$$\begin{aligned}I(w_n)&=\frac{1}{2}\|w_n\|^2-\int\limits_{\mathbb{R}^N}F\big(w_n^1\big)\,dx-\int\limits_{\mathbb{R}^N}G\big(w_n^2\big)\,dx\\&=\frac{1}{2}\|w_n\|^2+o(1)=2c+o(1).\end{aligned}\tag{2.6}$$

However, applying Lemma 2.1 with $t=\frac{2\sqrt{c}}{\|z_n\|}$, we have

$$I(w_n)\leqslant\frac{1+t^2}{2n}+I(z_n)\to c,$$

which contradicts to (2.6), so we ruled out the possibility of vanishing.

If nonvanishing occurs, then there exist $\eta>0$, $R<+\infty$ and $\{y_n\}\subset\mathbb{R}^N$ such that

$$\liminf_{n\to\infty}\int\limits_{B_R(y_n)}\big|w_n(x)\big|^2\,dx\geqslant\eta>0.\tag{2.7}$$

Let $\tilde u_n(x)=u_n(x-y_n)$, $\tilde v_n(x)=v_n(x-y_n)$ and $\tilde w_n(x)=w_n(x-y_n)=(\tilde w_n^1(x),\tilde w_n^2(x))$. Then $\|\tilde w_n\|=\|w_n\|=2\sqrt{c}$ and $\tilde z_n(x)=(\tilde u_n(x),\tilde v_n(x))$ with $\|\tilde z_n\|=\|z_n\|$. We may assume for some $\tilde w=(\tilde w^1,\tilde w^2)\in E$ that

$$\tilde w_n\rightharpoonup\tilde w\in E.$$

So

$$\tilde w_n\to\tilde w\quad\text{in }L^p_{loc}\big(\mathbb{R}^N\big)\text{ for }2\leqslant p<\frac{2N}{N-2},\qquad\tilde w_n\to\tilde w\quad\text{a.e. in }\mathbb{R}^N$$

and (2.7) implies that $\tilde w\neq 0$. For any $(\varphi,\psi)\in E$, let $\varphi_n(x)=\varphi(x+y_n)$, $\psi_n(x)=\psi(x+y_n)$, then by the fact that $I'(u_n,v_n)\to 0$ as $n\to\infty$ and $\|(\varphi_n,\psi_n)\|=\|(\varphi,\psi)\|$, we get

$$\begin{aligned}o(1)=&\int\limits_{\mathbb{R}^N}\big(\lambda\nabla\tilde w_n^1\cdot\nabla\varphi+\lambda\tilde w_n^1\varphi+\nabla\tilde w_n^1\cdot\nabla\psi+\tilde w_n^1\psi+\mu\nabla\tilde w_n^2\cdot\nabla\psi+\mu\tilde w_n^2\psi+\nabla\tilde w_n^2\cdot\nabla\varphi+\tilde w_n^2\varphi\big)\,dx\\&-t_n\int\limits_{\mathbb{R}^N}\big[\varphi f(\tilde u_n)+\psi g(\tilde v_n)\big]\,dx\end{aligned}\tag{2.8}$$

as $n\to\infty$. Let

$$p_n(x)=\begin{cases}\frac{f(\tilde u_n(x))}{\tilde u_n(x)}&\text{if }\tilde u_n(x)\neq 0;\\0&\text{if }\tilde u_n(x)=0\end{cases}$$

and

$$q_n(x)=\begin{cases}\frac{g(\tilde v_n(x))}{\tilde v_n(x)}&\text{if }\tilde v_n(x)\neq 0;\\0&\text{if }\tilde v_n(x)=0.\end{cases}$$

By (H_1)–(H_4), we see that

$$0 \leqslant p_n(x) \leqslant l, \qquad 0 \leqslant q_n(x) \leqslant m$$

for all $x \in \mathbb{R}^N$ and there exist two functions $p(x), q(x) \in L^\infty(\mathbb{R}^N) \cap L^2_{loc}(\mathbb{R}^N)$ such that

$$p_n \rightharpoonup p, \qquad q_n \rightharpoonup q \quad \text{in } L^2_{loc}(\mathbb{R}^N)$$

as $n \to \infty$. Hence

$$p_n(x)\tilde{w}_n^1 \rightharpoonup p(x)\tilde{w}^1(x), \qquad q_n(x)\tilde{w}_n^2 \rightharpoonup q(x)\tilde{w}^2(x) \quad \text{in } L^2(\mathbb{R}^N)$$

as $n \to \infty$. From (2.8), we have, for any $(\varphi, \psi) \in E$, that

$$\begin{aligned} o(1) = &\int_{\mathbb{R}^N} \left(\lambda \nabla \tilde{w}_n^1 \cdot \nabla \varphi + \lambda \tilde{w}_n^1 \varphi + \nabla \tilde{w}_n^1 \cdot \nabla \psi + \tilde{w}_n^1 \psi + \mu \nabla \tilde{w}_n^2 \cdot \nabla \psi + \mu \tilde{w}_n^2 \psi + \nabla \tilde{w}_n^2 \cdot \nabla \varphi + \tilde{w}_n^2 \varphi\right) dx \\ &- \int_{\mathbb{R}^N} p_n(x)\tilde{w}_n^1 \varphi\, dx - \int_{\mathbb{R}^N} q_n(x)\tilde{w}_n^2 \psi\, dx \end{aligned}$$

as $n \to \infty$. Letting $n \to \infty$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(\lambda \nabla \tilde{w}^1 \cdot \nabla \varphi + \lambda \tilde{w}^1 \varphi + \nabla \tilde{w}^1 \cdot \nabla \psi + \tilde{w}^1 \psi + \mu \nabla \tilde{w}^2 \cdot \nabla \psi + \mu \tilde{w}^2 \psi + \nabla \tilde{w}^2 \cdot \nabla \varphi + \tilde{w}^2 \varphi\right) dx \\ &\quad - \int_{\mathbb{R}^N} p(x)\tilde{w}^1 \varphi\, dx - \int_{\mathbb{R}^N} q(x)\tilde{w}^2 \psi\, dx = 0. \end{aligned} \tag{2.9}$$

Let

$$A_1 = \{x \in \mathbb{R}^N \colon \tilde{w}^1(x) \neq 0\}, \qquad A_2 = \{x \in \mathbb{R}^N \colon \tilde{w}^2(x) \neq 0\}.$$

Then $\lim_{n\to\infty} \tilde{u}_n(x) \to \infty$ if $x \in A_1$, $\lim_{n\to\infty} \tilde{v}_n(x) \to \infty$ if $x \in A_2$, and condition (H_3) implies that $p(x) = l$ for $x \in A_1$, $q(x) = m$ for $x \in A_2$. Hence, by (2.9), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(\lambda \nabla \tilde{w}^1 \cdot \nabla \varphi + \lambda \tilde{w}^1 \varphi + \nabla \tilde{w}^1 \cdot \nabla \psi + \tilde{w}^1 \psi + \mu \nabla \tilde{w}^2 \cdot \nabla \psi + \mu \tilde{w}^2 \psi + \nabla \tilde{w}^2 \cdot \nabla \varphi + \tilde{w}^2 \varphi\right) dx \\ &\quad = \int_{A_1} l\tilde{w}^1 \varphi\, dx + \int_{A_2} m\tilde{w}^2 \psi\, dx = \int_{\mathbb{R}^N} l\tilde{w}^1 \varphi\, dx - \int_{\mathbb{R}^N} m\tilde{w}^2 \psi\, dx. \end{aligned}$$

Therefore $\tilde{w} = (\tilde{w}^1, \tilde{w}^2) \neq 0$ is a solution pair of the linear system

$$\begin{cases} -(\Delta \tilde{w}^1 - \tilde{w}^1) - \mu(\Delta \tilde{w}^2 - \tilde{w}^2) = m\tilde{w}^2, \\ -(\Delta \tilde{w}^2 - \tilde{w}^2) - \lambda(\Delta \tilde{w}^1 - \tilde{w}^1) = l\tilde{w}^1, \end{cases} \quad x \in \mathbb{R}^N.$$

But Proposition 2.2 implies that $\tilde{w}^1 = \tilde{w}^2 = 0$, a contradiction. This completes the proof of the lemma. □

3. Proof of the main results

In this section, we prove our main results (Theorems 1.1 and 1.2). We keep the notations used in the previous sections.

Proof of Theorem 1.1. By Lemma 2.2, we know that the functional I defined in Theorem 1.2 possesses Mountain Pass geometry as described in Proposition 2.1. Proposition 2.1 implies that there exists a $(C)_c$-sequence $\{z_n\}$ for I, where $c > 0$. In terms of Lemma 2.3, $\{z_n\}$ is bounded in E. So we may assume that

$$z_n \rightharpoonup z = (u, v) \quad \text{in } E$$

as $n \to \infty$. According to the concentration-compactness principle [11], either vanishing or nonvanishing occurs for the concentration function of $|z_n|^2$. Denote $z_n = (u_n, v_n)$. If vanishing occurs, then for any $R > 0$,

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{y+B_R} \rho_n(x)\, dx = 0.$$

Using the vanishing lemma [11] and the fact that $\langle I'(z_n), z_n\rangle = o(1)$, we can show that $I(z_n) \to 0$ as $n \to \infty$. This contradicts to $I(z_n) \to c > 0$ as $n \to \infty$. Therefore, nonvanishing occurs. Then for some $\{y_n\} \subset \mathbb{R}^N$, $R > 0$ and $\eta > 0$, we have

$$\liminf_{n\to\infty} \int\limits_{B_R(y_n)} \left(|u_n|^2 + |v_n|^2\right) dx \geqslant \eta > 0.$$

Let $\tilde{z}_n(x) = z_n(x - y_n)$, then $\{\tilde{z}_n\}$ is also a $(C)_c$-sequence for I with

$$\tilde{z}_n \rightharpoonup \tilde{z} \quad \text{in } E$$

as $n \to \infty$ for some $\tilde{z} \in E$, and if we denote $\tilde{z}_n = (\tilde{u}_n, \tilde{v}_n)$, then

$$\liminf_{n\to\infty} \int\limits_{B_R(0)} \left(|\tilde{u}_n|^2 + |\tilde{v}_n|^2\right) dx \geqslant \eta > 0.$$

It implies $\tilde{z} \neq 0$. It is easy to see that $I'(\tilde{z}) = 0$ by the fact that $\{\tilde{z}_n\}$ is a $(C)_c$-sequence and $I'(\tilde{z}_n) \to 0$ as $n \to \infty$. Thus, $\tilde{z}$ is a nontrivial solution of (1.1). This completes the proof of Theorem 1.1. □

To prove Theorem 1.2, we first notice from Theorem 1.1 that the set

$$\left\{(u, v) \in E\colon (u, v) \text{ is a nontrivial solution pair of (1.1)}\right\}$$

is not an empty set. We define

$$I^\infty = \inf\left\{z \in E\colon I'(z) = 0,\ z \neq 0\right\}.$$

Then we have

Lemma 3.1. *If* (H_1)–(H_5) *hold, then* $I^\infty > 0$.

Proof. By Theorem 1.1, we know that

$$\left\{z \in E\colon I'(z) = 0,\ z \neq 0\right\} \neq \emptyset,$$

so I^∞ is finite. Let $z = (u, v)$ be a solution of (1.1), then $\langle I'(z), z\rangle = 0$. It yields

$$I(z) = I(u, v) = \int\limits_{\mathbb{R}^N} \left(\frac{1}{2} f(u)u - F(u)\right) dx + \int\limits_{\mathbb{R}^N} \left(\frac{1}{2} g(v)v - G(v)\right) dx \geqslant 0$$

by (H_5). Since z is arbitrary, so $I^\infty \geqslant 0$.

Now we show that I^∞ is assumed and positive. We suppose, by contradiction, that $I^\infty = 0$. Suppose now that $z_n = (u_n, v_n)$ is a minimizing sequence of I^∞, then

$$I(z_n) \to I^\infty = 0, \qquad I'(z_n) = 0, \quad z_n \neq 0. \tag{3.1}$$

Thus $\langle I'(z_n), z_n\rangle = 0$, that is

$$\|z_n\|^2 = \int\limits_{\mathbb{R}^N} f(u_n)u_n\, dx + \int\limits_{\mathbb{R}^N} g(v_n)v_n\, dx. \tag{3.2}$$

First of all, we show that $\{z_n\}$ is bounded in E. By contradiction, we suppose that

$$\|z_n\| \to \infty$$

as $n \to \infty$ and for any fixed $\alpha > 0$, let

$$t_n = \frac{\alpha}{\|z_n\|}, \qquad w_n = t_n z_n = (t_n u_n, t_n v_n) \triangleq \left(w_n^1, w_n^2\right).$$

Clearly, $\{w_n\}$ is bounded in E. For $\rho_n(x) = |w_n(x)|^2 = t_n^2(u_n^2 + v_n^2)$, by the concentration-compactness principle in [11] too, we know that either vanishing or nonvanishing happens, we will get contradiction in both cases.

If vanishing happens, by the same process as in giving (2.6) and noticing that $\|w_n\| = \alpha > 0$, we have

$$I(w_n) = \frac{1}{2}\alpha^2 + o(1). \tag{3.3}$$

But, by Lemma 2.1 and (3.2),

$$I(w_n) = I(t_n z_n) \leqslant I(z_n) \to I^\infty = 0, \tag{3.4}$$

which is impossible by (3.3).

If nonvanishing occurs, then there exist $\eta > 0$, $R < +\infty$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int\limits_{y_n+B_R} |w_n(x)|^2\,dx \geqslant \eta > 0. \tag{3.5}$$

Set $\tilde{w}_n(x) = w_n(x - y_n) = (\tilde{w}_n^1(x), \tilde{w}_n^2(x))$. Then $\|\tilde{w}_n\| = \|w_n\| = \alpha$. By Sobolev embedding, we may assume that for some $\tilde{w} = (\tilde{w}^1, \tilde{w}^2) \in E$ that

$$\tilde{w}_n \rightharpoonup \tilde{w} \in E.$$

So

$$\tilde{w}_n \to \tilde{w} \quad \text{in } L^p_{loc}(\mathbb{R}^N) \quad \text{for } 2 \leqslant p < \frac{2N}{N-2}, \qquad \tilde{w}_n \to \tilde{w} \quad \text{a.e. in } \mathbb{R}^N.$$

These and (3.5) imply that

$$\tilde{w} \neq 0. \tag{3.6}$$

By condition (H_4), for n large enough, we have

$$t_n = \frac{\alpha}{\|z_n\|} \in (0,1) \quad \text{and} \quad \frac{f(t_n u_n)}{t_n u_n} \leqslant \frac{f(u_n)}{u_n}, \qquad \frac{g(t_n v_n)}{t_n v_n} \leqslant \frac{g(v_n)}{v_n}.$$

Hence by (3.2), we obtain

$$\begin{aligned}
&\|w_n\|^2 - \int\limits_{\mathbb{R}^N} f(w_n^1) w_n^1\,dx - \int\limits_{\mathbb{R}^N} g(w_n^2) w_n^2\,dx \\
&= t_n^2 \Bigg[\|z_n\|^2 - \int\limits_{\mathbb{R}^N} \frac{f(t_n u_n)}{t_n u_n} u_n^2\,dx - \int\limits_{\mathbb{R}^N} \frac{g(t_n v_n)}{t_n v_n} v_n^2\,dx \Bigg] \\
&\geqslant t_n^2 \Bigg[\|z_n\|^2 - \int\limits_{\mathbb{R}^N} \frac{f(u_n)}{u_n} u_n^2\,dx - \int\limits_{\mathbb{R}^N} \frac{g(v_n)}{v_n} v_n^2\,dx \Bigg] = 0.
\end{aligned}$$

Then it follows from (H_4) and Fatou's lemma that

$$\begin{aligned}
I(w_n) &= \frac{1}{2}\|w_n\|^2 - \int\limits_{\mathbb{R}^N} F(w_n^1)\,dx - \int\limits_{\mathbb{R}^N} G(w_n^2)\,dx \\
&\geqslant \int\limits_{\mathbb{R}^N} \Big[\frac{1}{2} f(w_n^1) w_n^1 - F(w_n^1)\Big]\,dx + \int\limits_{\mathbb{R}^N} \Big[\frac{1}{2} g(w_n^2) w_n^2 - G(w_n^2)\Big]\,dx \\
&= \int\limits_{\mathbb{R}^N} \Big[\frac{1}{2} f(\tilde{w}_n^1) \tilde{w}_n^1 - F(\tilde{w}_n^1)\Big]\,dx + \int\limits_{\mathbb{R}^N} \Big[\frac{1}{2} g(\tilde{w}_n^2) \tilde{w}_n^2 - G(\tilde{w}_n^2)\Big]\,dx \\
&\geqslant \int\limits_{\mathbb{R}^N} \Big[\frac{1}{2} f(\tilde{w}^1) \tilde{w}^1 - F(\tilde{w}^1)\Big]\,dx + \int\limits_{\mathbb{R}^N} \Big[\frac{1}{2} g(\tilde{w}^2) \tilde{w}^2 - G(\tilde{w}^2)\Big]\,dx + o(1).
\end{aligned}$$

So by (H_5), (3.4) and (3.6), we get

$$0 < \int\limits_{\mathbb{R}^N} \Big[\frac{1}{2} f(\tilde{w}^1) \tilde{w}^1 - F(\tilde{w}^1)\Big]\,dx + \int\limits_{\mathbb{R}^N} \Big[\frac{1}{2} g(\tilde{w}^2) \tilde{w}^2 - G(\tilde{w}^2)\Big]\,dx \leqslant 0,$$

which is impossible. Thus $\{z_n\}$ is bounded in E.

Next, let $\rho_n(x) = |z_n(x)|^2 = u_n^2 + v_n^2$, then either vanishing or nonvanishing holds for $\rho_n(x)$ by the concentration-compactness principle [11].

If vanishing occurs, similar to (2.6), we have

$$I(z_n) = \frac{1}{2}\|z_n\|^2 + o(1). \tag{3.7}$$

From (2.1) and (3.2), we have

$$\|z_n\|^2 = \int_{\mathbb{R}^N} \left[f(u_n)u_n + g(v_n)v_n \right] dx$$
$$\leqslant \epsilon |u_n|_2^2 + C_\epsilon |u_n|_p^p + \epsilon |v_n|_2^2 + C_\epsilon |v_n|_p^p$$
$$\leqslant c\varepsilon \|z_n\|^2 + C_\epsilon \|z_n\|^p,$$

which means that there is a $\delta > 0$ such that

$$\|z_n\| \geqslant \delta \tag{3.8}$$

when ϵ is small enough. But, $I(z_n) \to I^\infty = 0$, then (3.7) and (3.8) are contradictory.

If nonvanishing occurs, then there exist $\eta > 0$, $R < +\infty$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{y_n+B_R} \left(|u_n|^2 + |v_n|^2\right) dx \geqslant \eta > 0. \tag{3.9}$$

Let $\tilde{z}_n(x) = z_n(x - y_n) = (u_n(x - y_n), v_n(x - y_n)) = (\tilde{u}_n(x), \tilde{v}_n(x))$. Then $\|\tilde{z}_n\| = \|z_n\|$ and $\{\tilde{z}_n\}$ is bounded in E. So by Sobolev embedding, we may assume that for some $\tilde{z} = (\tilde{u}, \tilde{v}) \in E$,

$$\tilde{z}_n \to \tilde{z} \quad \text{a.e. in } \mathbb{R}^N$$

and $\tilde{z}(x) \not\equiv 0$ by (3.9). Using again (H_5), (3.1), (3.2) and Fatou's lemma, we have

$$0 < \int_{\mathbb{R}^N} \left[\frac{1}{2} f(\tilde{u})\tilde{u} - F(\tilde{u}) \right] dx + \int_{\mathbb{R}^N} \left[\frac{1}{2} g(\tilde{v})\tilde{v} - G(\tilde{v}) \right] dx$$
$$\leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{2} f(\tilde{u}_n)\tilde{u}_n - F(\tilde{u}_n) + \frac{1}{2} g(\tilde{v}_n)\tilde{v}_n - G(\tilde{v}_n) \right] dx$$
$$= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{1}{2} f(u_n)u_n - F(u_n) + \frac{1}{2} g(v_n)v_n - G(v_n) \right] dx$$
$$= I^\infty = 0,$$

which means that nonvanishing is neither impossible. So $I^\infty > 0$. □

The proof of Theorem 1.2 will be completed by the following lemma.

Lemma 3.2. *Assume* (H_1)–(H_5) *hold. If* $\lambda\mu > 1$, *then* I^∞ *is assumed.*

Proof. By (1.1) and (2.1), we have

$$\int_{\mathbb{R}^N} \left[\nabla u \cdot \nabla v + uv + \mu\left(|\nabla v|^2 + |v|^2\right) \right] dx = \int_{\mathbb{R}^N} g(v)v\, dx \leqslant \int_{\mathbb{R}^N} \left(\epsilon |v|^2 + c_\epsilon |v|^p\right) dx.$$

Similarly,

$$\int_{\mathbb{R}^N} \left[\nabla u \cdot \nabla v + uv + \lambda\left(|\nabla u|^2 + |u|^2\right) \right] dx \leqslant \int_{\mathbb{R}^N} \left(\epsilon |u|^2 + c_\epsilon |u|^p\right) dx.$$

Adding the above two inequalities we obtain

$$\|z\|^2 = \int_{\mathbb{R}^N} \left[\lambda\left(|\nabla u|^2 + u^2\right) + 2(\nabla u \cdot \nabla v + uv) + \mu\left(|\nabla v|^2 + v^2\right) \right] dx$$
$$\leqslant \epsilon \int_{\mathbb{R}^N} \left(|u|^2 + |v|^2\right) dx + c_\epsilon \int_{\mathbb{R}^N} \left(|u|^p + |v|^p\right) dx$$
$$\leqslant \epsilon\left(\|u\|_{H^1(\mathbb{R}^N)}^2 + \|v\|_{H^1(\mathbb{R}^N)}^2\right) + c'_\epsilon\left(\|u\|_{H^1(\mathbb{R}^N)}^2 + \|v\|_{H^1(\mathbb{R}^N)}^2\right)^{\frac{p}{2}}.$$

Since the norms $\|\cdot\|$ and $\|\cdot\|_E$ are equivalent if $\lambda\mu > 1$ by (1.3), so choosing $\epsilon > 0$ properly, we obtain

$$\|z\| \geqslant c > 0. \tag{3.10}$$

Suppose now that $z_n = (u_n, v_n)$ is a minimizing sequence of $I^\infty > 0$, that is,

$$I(z_n) \to I^\infty, \qquad I'(z_n) = 0, \quad z_n \neq 0.$$

Lemma 2.3 implies that $\{z_n\}$ is uniformly bounded in E. Hence, we may assume

$$z_n \rightharpoonup z = (u, v) \quad \text{in } E, \qquad z_n \to z \quad \text{a.e. in } \mathbb{R}^N, \qquad z_n \to z \quad \text{in } L^q_{loc}(\mathbb{R}^N) \times L^q_{loc}(\mathbb{R}^N)$$

as $n \to \infty$ for any $2 \leqslant q < 2N/(N-2)$. Now, for the concentration function of $|u_n|^2 + |v_n|^2$, by the concentration-compactness principle [11], we know that either vanishing or nonvanishing happens.

If vanishing happens, by the vanishing lemma in [11], we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^q \, dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^q \, dx = 0$$

for $2 < q < 2N/(N-2)$. Since $I'(z_n) = 0$, then $\langle I'(z_n), (0, v_n)\rangle = 0$. Noting that $\|z_n\|$ is bounded in E, then by (2.1), we get

$$\begin{aligned}
\langle I'(z_n), (0, v_n)\rangle &= \int_{\mathbb{R}^N} \left[\nabla u_n \cdot \nabla v_n + u_n v_n + \mu\left(|\nabla v_n|^2 + |v_n|^2\right)\right] dx \\
&= \int_{\mathbb{R}^N} g(v_n) v_n \, dx \leqslant \int_{\mathbb{R}^N} \left(\epsilon |v_n|^2 + c_\epsilon |v_n|^p\right) dx \\
&\leqslant \epsilon c + c_\epsilon o(1).
\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^N} \left[\nabla u_n \cdot \nabla v_n + u_n v_n + \lambda\left(|\nabla u_n|^2 + |u_n|^2\right)\right] dx \leqslant \epsilon c + c_\epsilon o(1).$$

Adding the above two inequalities, we obtain

$$\|z_n\|^2 = \int_{\mathbb{R}^N} \left[\lambda\left(|\nabla u_n|^2 + u_n^2\right) + 2(\nabla u_n \cdot \nabla v_n + u_n v_n) + \mu\left(|\nabla v_n|^2 + v_n^2\right)\right] dx \leqslant \epsilon c + c_\epsilon o(1).$$

So by (3.10),

$$0 < c^2 \leqslant \|z_n\|^2 \leqslant \epsilon c + c_\epsilon o(1).$$

Letting $n \to \infty$ and $\epsilon \to 0$ we obtain a contradiction. Thus, the vanishing was ruled out and nonvanishing occurs, that is, there exist $\eta > 0$, $R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \inf_{y \in \mathbb{R}^N} \int_{B_R(y_n)} \left(|u_n|^2 + |v_n|^2\right) dx \geqslant \eta > 0.$$

Let $\tilde{z}_n = z_n(x - y_n)$. Then,

$$\liminf_{n\to\infty} \int_{B_R(0)} \left(|\tilde{u}_n|^2 + |\tilde{v}_n|^2\right) dx \geqslant \eta > 0, \tag{3.11}$$

and $I(\tilde{z}_n) = I(z_n) \to I^\infty$ as $n \to \infty$, $I'(\tilde{z}_n) = 0$ as well as

$$\|\tilde{z}_n\| = \|z_n\|, \qquad \tilde{z}_n \rightharpoonup \tilde{z} = (\tilde{u}, \tilde{v}) \quad \text{in } E.$$

Obviously, $I'(\tilde{z}) = 0$. (3.11) implies $\tilde{z} \neq 0$ and $\tilde{z}_n \to \tilde{z}$ a.e. in $\mathbb{R}^N$. By (H_5) and Fatou's lemma,

$$\begin{aligned}
I^\infty &= \lim_{n\to\infty} I(z_n) = \lim_{n\to\infty} I(\tilde{z}_n) \\
&\geqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} \left[\left(\frac{1}{2} f(\tilde{u}_n)\tilde{u}_n - F(\tilde{u}_n)\right) + \left(\frac{1}{2} g(\tilde{v}_n)\tilde{v}_n - G(\tilde{v}_n)\right)\right] dx
\end{aligned}$$

$$\geqslant \int\limits_{\mathbb{R}^N} \left[\left(\frac{1}{2} f(\tilde{u})\tilde{v} - F(\tilde{v}) \right) + \left(\frac{1}{2} g(\tilde{v})\tilde{v} - G(\tilde{v}) \right) \right] dx$$

$$= I(\tilde{z}).$$

Consequently, I^∞ is assumed by $\tilde{z} \in E \setminus \{0\}$. The proof is complete. □

Proof of Theorem 1.2. This is a direct consequence of Lemmas 3.1 and 3.2. □

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